

Phase space analysis for three and four massive particles in final states

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We propose formulae for computing the phase space integrals of $1 \rightarrow 3$ and $1 \rightarrow 4$ processes with massive particles in final states. As an application of these formulae we study the final state mass effects in some interesting phenomenological cases, giving fully integrated analytic results for the corresponding phase spaces. We consider also the $B_s - \bar{B}_s$ process at NNLO and calculate one of the most complicated master integrals, which contributes to the $\Delta\Gamma_{B_s}$ at $O(\alpha_s^2)$.

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I. INTRODUCTION

The $B_{s,d}$ -meson rare decays and oscillations are known to be a unique source of indirect information about physics at scales of several hundreds GeV. In the Standard Model all these processes proceed through loop diagrams and thus are relatively suppressed. In the extensions of the Standard Model the contributions stemming from the diagrams with "new" particles in the loops can be comparable or even larger than the contribution from the SM. Thus getting experimental information about rare decays puts strong constraints on the extensions of the SM or, if we are lucky, can lead to a strong disagreement with the SM predictions providing an evidence of some "new physics". To make a rigorous comparison between experiment and theory, one has to get refined theoretical predictions for the rare decay at hand. In particular, perturbative QCD corrections through next-to-next-to leading order (NNLO) in α_s are needed. When calculating higher order QCD corrections, along with the calculation of virtual corrections, it is necessary to take into account real emission of gluons and quark-antiquark pairs. Then the phase space integrals contain infrared and collinear singularities, which can be regularized using dimensional regularization scheme.

Sometimes however it is more convenient to introduce a small mass as an infrared regulator which can regularize both the soft and collinear divergences. This can be for instance massive s quark for $b \rightarrow s\gamma$ [1] and $b \rightarrow s\gamma\gamma$ [2] or massive light quarks for $b \rightarrow s\bar{q}q\gamma$ ($q = u, d, s$) [3]. Another example is the calculation of CP-asymmetry and width difference- $\Delta\Gamma$ in $B_{s,d} - \bar{B}_{s,d}$ mixing at NLO and NNLO in α_s , when the calculation gets simplified by the usage of a small gluon mass as an infrared regulator.

In this article we derive formulae for the $1 \rightarrow 3$ [4] and $1 \rightarrow 4$ phase spaces, where the final state particles are massive. Following the approach of [5] for the $1 \rightarrow 3$ process we derive formulae for the phase space in the rest frame of two particles in the final state and obtain a factorized parameterization of the phase-space for the case when all three particles in final state are massive. In case of $1 \rightarrow 4$ process we consider similar method, deriving the phase space formula in the rest frame of three particles in the final state. Here we come to the factorized parameterization of phase-space when only one

particle in the final state is massive.

The paper is organized as follows. In Section II we discuss three-particle phase space. Section III is devoted to the investigation of four-particle phase space. In Section IV we apply our technique to the calculation of master integral connected with the $B - \bar{B}$ mixing at $O(\alpha_s^2)$. The Appendix contains formulae of $1 \rightarrow 3$ and $1 \rightarrow 4$ integrated phase spaces for some particular cases.

II. THREE-PARTICLE PHASE SPACE

Here we consider the three particle decay, when particle with momentum p' decays into three particles with momenta p_i and masses m_i , $i = 1, 2, 3$; $p' = p_1 + p_2 + p_3$. We start from the well-known expression for the differential decay width:

$$d\Gamma = \frac{1}{2m} |\overline{M}|^2 D\Phi,$$

where $|\overline{M}|^2$ is the squared matrix element, summed and averaged over spins and colors of the particles in the final and initial states respectively and m is the mass of decaying particle.

The phase space formula in the rest frame of particles 1 and 3 with momenta p'_1 and p_3 was derived in [5]:

$$D\Phi = m^{2(d-3)} D\Phi_1 D\Phi_2 ds_{13}, \quad (2.1)$$

$$D\Phi_1 = (2\pi)^d s_{13}^{(d-2)/2} \frac{d^{d-1}p}{2p^0} \frac{d^{d-1}p_2}{(2\pi)^{d-1}2p_2^0} \delta^d(p - p_2 - q),$$

$$D\Phi_2 = \frac{d^{d-1}p_1}{(2\pi)^{d-1}2p_1^0} \frac{d^{d-1}p_3}{(2\pi)^{d-1}2p_3^0} \delta^d(q - p_1 - p_3).$$

where $d = 4 - 2\epsilon$ and "dimensionless" momenta $p = p'/m$, $p_i = p'_i/m$, $i=1,2,3$ are introduced. In (2.1) we introduced an additional integration over $s_{13} = q^2$, $q = p_1 + p_3$ (in the considered frame $q = (\sqrt{s_{13}}, \vec{0})$). Now we can integrate over $d-1$ components of p_1 and p_2 using the spatial parts of two d -dimensional δ functions. To carry out the remaining integrations we choose the coordinate axes in a way that particle momenta have the following components:

$$\begin{aligned} p &= (E, |\vec{p}|, 0, 0, \dots), \\ p_3 &= (E_3, |\vec{p}_3| \cos \vartheta, |\vec{p}_3| \sin \vartheta, 0, \dots), \end{aligned} \quad (2.2)$$

where the dots correspond to the components of extra space dimensions, which are all zero. Making use of the remaining two one-dimensional δ -functions we express E and E_3 in the following way:

$$E = \frac{1 + s_{13} - x_2}{2\sqrt{s_{13}}} \quad ; \quad E_3 = \frac{x_3 + s_{13} - x_1}{2\sqrt{s_{13}}}, \quad (2.3)$$

$$\begin{aligned} D\Phi(1 \rightarrow 3) &= m^{2(d-3)} \frac{\Omega_{d-1} \Omega_{d-2}}{2^4 (2\pi)^{2d-3}} s_{13}^{(d-4)/2} (|\vec{p}| |\vec{p}_3| \sin \vartheta)^{d-3} d\vartheta ds_{13} \\ &= m^{2(d-3)} \frac{\Omega_{d-1} \Omega_{d-2}}{2^4 (2\pi)^{2d-3}} s_{13}^{(d-4)/2} (|\vec{p}| |\vec{p}_3|)^{d-3} (1 - \cos^2 \vartheta)^{\frac{d-4}{2}} d\cos \vartheta ds_{13} \end{aligned} \quad (2.4)$$

where Ω_d is the solid angle in d dimensions,

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad (2.5)$$

and

$$\begin{aligned} |\vec{p}| &= \frac{1}{2\sqrt{s_{13}}} \sqrt{(1 + x_2 - s_{13})^2 - 4x_2}, \\ |\vec{p}_3| &= \frac{1}{2\sqrt{s_{13}}} \sqrt{(x_1 + x_3 - s_{13})^2 - 4x_1 x_3} \end{aligned} \quad (2.6)$$

are 3-momenta expressed by *Källén* functions.

For the integration limits we have

$$\begin{aligned} (\sqrt{x_1} + \sqrt{x_3})^2 &\leq s_{13} \leq (1 - \sqrt{x_2})^2, \\ -1 &\leq \cos \vartheta \leq 1. \end{aligned} \quad (2.7)$$

We represent all scalar products via variables s_{13} and $\cos \vartheta$:

$$\begin{aligned} p \cdot p_2 &= \frac{1 + x_2 - s_{13}}{2} \\ p \cdot p_1 &= \frac{2(1 + x_1 + x_3) - x_2 - X}{4} \end{aligned}$$

where $x_i = m_i^2/m^2$, $i = 1, 2, 3$ are dimensionless variables. After integration over those angles on which $|\vec{M}|^2$ does not depend, we obtain a factorized formula for the three massive particles phase space (unlike [4], where the corresponding formula is not factorized):

$$\begin{aligned} p \cdot p_3 &= \frac{-2(x_1 + x_3) - x_2 + 2s_{13} + X}{4} \\ p_2 \cdot p_1 &= \frac{2 + 4x_3 - x_2 - 2s_{13} - X}{4} \\ p_2 \cdot p_3 &= \frac{-x_2 - 4x_3 + X}{4} \\ p_1 \cdot p_3 &= \frac{s_{13} - x_1 - x_3}{2} \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} X &= -4|\vec{p}| |\vec{p}_3| \cos \vartheta + 1 + 2(x_1 + x_3) - s_{13} \\ &\quad - \frac{(1 + s_{13} - x_2)(x_1 - x_3)}{s_{13}}, \end{aligned} \quad (2.9)$$

and $|\vec{p}|$ and $|\vec{p}_3|$ are defined in (2.6).

From the formula (2.4) for the three-particle phase space one can get a simpler formula assuming that the masses of two particles in the final state are equal to zero: $m_2 = m_3 = 0$. In that case we make the replacements: $\cos \vartheta = 2\lambda_1 - 1$ and $s_{13} = \lambda_2(1 - x_1) + x_1$, where $0 < \lambda_{1,2} < 1$. Then we get:

$$\begin{aligned} D\Phi(1 \rightarrow 3) &= m^{2(d-3)} \frac{\Omega_{d-1} \Omega_{d-2}}{2^{d+1} (2\pi)^{2d-3}} d\lambda_1 d\lambda_2 \\ &\quad \times (1 - x_1)^{2d-5} [\lambda_2(1 - x_1) + x_1]^{(2-d)/2} [\lambda_2(1 - \lambda_2)]^{d-3} [\lambda_1(1 - \lambda_1)]^{\frac{d-4}{2}} \end{aligned} \quad (2.10)$$

If we put $x_1 = 0$ (i.e. $m_1 = 0$) in the expression (2.10), we get the formula (2) of [6].

In the Appendix A we study the final state mass effects in some interesting phenomenological cases, providing fully integrated analytic formulae for the $1 \rightarrow 3$ processes.

III. FOUR-PARTICLE PHASE SPACE

Now we proceed to the case of four particle decay. For this case one can derive the expression for the phase space in the similar way as in [5]. The only difference is that instead of two particle's rest frame introduced in [5] here the rest frame of three particles is used (see [7]). We

consider a particle with momentum p' , which decays into four particles with momenta p'_i and masses m_i , $i = 1, \dots, 4$ and $p' = p'_1 + p'_2 + p'_3 + p'_4$. Here also we introduce "dimensionless" momenta $p = p'/m$, $p_i = p'_i/m$, $i=1, \dots, 4$. In the rest frame of particles 2, 3 and 4 with momenta p_2, p_3, p_4 (where $\vec{p}_2 + \vec{p}_3 + \vec{p}_4 = 0$) four-particle phase space can be written in the following way

$$\begin{aligned} d\Phi &= m^{3d-8} (2\pi)^d d\Phi_1 d\Phi_2 ds_{234}, \\ d\Phi_1 &= (s_{234})^{(d-2)/2} \frac{d^{d-1}p}{2p^0} \frac{d^{d-1}p_1}{(2\pi)^{d-1}2p_1^0} \delta^d(p - p_1 - q), \\ d\Phi_2 &= \frac{d^{d-1}p_2}{(2\pi)^{d-1}2p_2^0} \frac{d^{d-1}p_3}{(2\pi)^{d-1}2p_3^0} \frac{d^{d-1}p_4}{(2\pi)^{d-1}2p_4^0} \\ &\quad \times \delta^d(q - p_2 - p_3 - p_4), \end{aligned} \quad (3.1)$$

where $s_{234} = (p_2 + p_3 + p_4)^2$ and q is a d -dimensional vector with components $(\sqrt{s_{234}}, \vec{0})$ in the rest frame of particles 2, 3 and 4. We choose the coordinate axes in such a way, that the particle momenta have the following com-

ponents:

$$\begin{aligned} p_2 &= (E_2, |\vec{p}_2|, 0, 0, \dots), \\ p_1 &= (E_1, |\vec{p}_1| \cos \vartheta_1, |\vec{p}_1| \sin \vartheta_1, 0, \dots), \\ p_3 &= (E_3, |\vec{p}_3| \cos \vartheta_3, |\vec{p}_3| \sin \vartheta_3 \cos \phi_3, |\vec{p}_3| \sin \vartheta_3 \sin \phi_3, \dots), \end{aligned} \quad (3.2)$$

where the dots, as before, correspond to the components of extra space dimensions, which are all zero. We can immediately integrate over those angles on which none of the scalar products depends and obtain

$$\begin{aligned} d^{d-1}p_2 &= \Omega_{d-1} |\vec{p}_2|^{d-2} d|\vec{p}_2|, \\ d^{d-1}p_1 &= \Omega_{d-2} (\sin \vartheta_1)^{d-4} d\cos \vartheta_1 |\vec{p}_1|^{d-2} d|\vec{p}_1|, \\ d^{d-1}p_3 &= \Omega_{d-3} (\sin \phi_3)^{d-5} d\cos \phi_3 (\sin \vartheta_3)^{d-4} \\ &\quad \times d\cos \vartheta_3 |\vec{p}_3|^{d-2} d|\vec{p}_3|. \end{aligned} \quad (3.3)$$

p_4 and p can be integrated out immediately using the spatial parts of the δ functions in (3.1). Thus for the phase space we get

$$D\Phi_1 = \frac{(s_{234})^{(d-2)/2}}{2E} \frac{\Omega_{d-2} (\sin \vartheta_1)^{d-4} d\cos \vartheta_1 |\vec{p}_1|^{d-2} d|\vec{p}_1|}{(2\pi)^{d-1}2E_1} \delta(E - E_1 - \sqrt{s_{234}}), \quad (3.4)$$

$$D\Phi_2 = \frac{\Omega_{d-1} |\vec{p}_2|^{d-2} d|\vec{p}_2|}{(2\pi)^{d-1}2E_2} \frac{\Omega_{d-3} (\sin \phi_3)^{d-5} d\cos \phi_3 (\sin \vartheta_3)^{d-4} d\cos \vartheta_3 |\vec{p}_3|^{d-2} d|\vec{p}_3|}{(2\pi)^{d-1}2E_3} \frac{\delta(\sqrt{s_{234}} - E_2 - E_3 - E_4)}{(2\pi)^{d-1}2E_4}, \quad (3.5)$$

where

$$E_4 = \sqrt{|\vec{p}_2|^2 + |\vec{p}_3|^2 + 2|\vec{p}_2||\vec{p}_3| \cos \vartheta_3 + x_4}, \quad (3.6)$$

$$E = \sqrt{|\vec{p}_1|^2 + 1}, \quad (3.7)$$

$$x_i = m_i^2/m^2, i = 1, \dots, 4$$

where $|\vec{p}_i| = \sqrt{E_i^2 - x_i}$, $i = 1, 2, 3$. Making use of the remaining two one-dimensional δ -functions, we can express E_1 and E_3 in terms of the other variables. Using the δ

function in $d\Phi_1$ and (3.7) we get:

$$E_1 = \frac{1 - x_1 - s_{234}}{2\sqrt{s_{234}}}, \quad E = \frac{1 - x_1 + s_{234}}{2\sqrt{s_{234}}}. \quad (3.8)$$

From the δ function in $d\Phi_2$ together with (3.6) we find the expression for E_3

$$\begin{aligned} E_3 &= \frac{1}{-2(\sqrt{s_{234}} - E_2)^2 + 2|\vec{p}_2|^2 \cos^2 \vartheta_3} [(E_2 - \sqrt{s_{234}})(s_{234} - 2E_2\sqrt{s_{234}} + x_2 + x_3 - x_4) \\ &\quad + |\vec{p}_2| \cos \vartheta_3 \sqrt{(s_{234} - 2E_2\sqrt{s_{234}} + x_2 - x_3 - x_4)^2 - 4x_3(x_4 + |\vec{p}_2|^2 \sin^2 \vartheta_3)}] . \end{aligned} \quad (3.9)$$

Finally, introducing new variables: $z_1 = \cos \vartheta_1$, $z_{31} = \cos \vartheta_3$, $z_{32} = \cos \phi_3$, we get for the four-particle phase space:

$$\begin{aligned} D\Phi(1 \rightarrow 4) &= \frac{m^{3d-8}}{32} \frac{\Omega_{d-1}\Omega_{d-2}\Omega_{d-3}}{(2\pi)^{3d-4}} dz_1 dz_{31} dz_{32} dE_2 ds_{234} \\ &\quad \times (1 - z_1^2)^{\frac{(d-4)}{2}} (1 - z_{31}^2)^{\frac{(d-4)}{2}} (1 - z_{32}^2)^{\frac{(d-5)}{2}} \frac{W^{d-3}}{\sqrt{s_{234}} - E_2 + \frac{E_3}{|\vec{p}_3|} |\vec{p}_2| z_{31}} \end{aligned} \quad (3.10)$$

where

$$W = |\vec{p}_1| |\vec{p}_2| |\vec{p}_3| \sqrt{s_{234}}. \quad (3.11)$$

The integration variables $s_{234}, E_2, z_1, z_{31}, z_{32}$ in (3.10) have the following limits

$$(\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4})^2 \leq s_{234} \leq (1 - \sqrt{x_1})^2, \quad (3.12)$$

$$\sqrt{x_2} \leq E_2 \leq \frac{s_{234} + x_2 - (\sqrt{x_3} + \sqrt{x_4})^2}{2\sqrt{s_{234}}},$$

$$-1 \leq z_i \leq 1. \quad (3.13)$$

Different possible scalar products between 4-momenta of the particles can be obtained automatically from the expressions for their components (3.2) using the conservation of momentum. All the scalar products can be expressed through the 5 integration variables in (3.10):

$$\begin{aligned} p_1 \cdot p_2 &= E_1 E_2 - |\vec{p}_1| |\vec{p}_2| \cos \vartheta_1, \\ p_2 \cdot p_3 &= E_2 E_3 - |\vec{p}_2| |\vec{p}_3| \cos \vartheta_3, \\ p_1 \cdot p_3 &= E_1 E_3 - |\vec{p}_1| |\vec{p}_3| (\cos \vartheta_1 \cos \vartheta_3 \\ &\quad + \sin \vartheta_1 \sin \vartheta_3 \cos \phi_3), \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} p_1 \cdot p_4 &= E_1 \sqrt{s_{234}} - p_1 \cdot p_3 - p_1 \cdot p_2, \\ p_2 \cdot p_4 &= E_2 \sqrt{s_{234}} - x_2 - p_2 \cdot p_3, \\ p_3 \cdot p_4 &= E_3 \sqrt{s_{234}} - x_3 - p_2 \cdot p_3, \end{aligned} \quad (3.15)$$

In right-hand side of (3.15) the expressions given in (3.14) should be used. Note that in (3.14) and (3.15) E_1 and E_3 are determined by formulae (3.8) and (3.9).

We can also obtain a factorized formula for the 4 particle phase space with 1 massive and 3 massless particles in the final state, taking $x_2 = x_3 = x_4 = 0$. Making the following substitutions in (3.10)

$$\begin{aligned} s_{234} &= (1 - \sqrt{x_1})^2 \lambda_1, \quad E_2 = \frac{1}{2} \sqrt{\lambda_1} (1 - \lambda_2) (1 - \sqrt{x_1}), \\ z_{31} &= \frac{\lambda_2 (1 - \lambda_4) - \lambda_4}{\lambda_2 (1 - \lambda_4) + \lambda_4}, \quad z_1 = 2\lambda_3 - 1, \\ z_{32} &= 2\lambda_5 - 1, \end{aligned} \quad (3.16)$$

the differential phase space can be expressed in the following form:

$$\begin{aligned} D\Phi(1 \rightarrow 4) &= \frac{m^{3d-8}}{128} \frac{\Omega_{d-1} \Omega_{d-2} \Omega_{d-3}}{(2\pi)^{3d-4}} d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 d\lambda_5 \\ &\times (1 - \sqrt{x_1})^{3d-7} [(1 - \lambda_1)((1 + \sqrt{x_1})^2 - \lambda_1(1 - \sqrt{x_1})^2)]^{(d-3)/2} \\ &\times [\lambda_1(1 - \lambda_2)]^{d-3} [\lambda_2(1 - \lambda_3)\lambda_3(1 - \lambda_4)\lambda_4]^{(d-4)/2} [\lambda_5(1 - \lambda_5)]^{(d-5)/2}. \end{aligned} \quad (3.17)$$

where $\lambda_i = [0, 1]$, $i = 1 \dots 5$. The corresponding scalar products in terms of the λ_i will have the following form

$$\begin{aligned} s_{234} &= (p_2 + p_3 + p_4)^2 = (1 - \sqrt{x_1})^2 \lambda_1, \\ s_{34} &= (p_3 + p_4)^2 = (1 - \sqrt{x_1})^2 \lambda_1 \lambda_2, \\ s_{23} &= (p_2 + p_3)^2 = (1 - \sqrt{x_1})^2 \lambda_1 (1 - \lambda_2) \lambda_4, \end{aligned}$$

where

$$\begin{aligned} s_{134} &= (p_1 + p_3 + p_4)^2 = (s_{134}^+ - s_{134}^-) \lambda_3 + s_{134}^-, \\ s_{13} &= (p_1 + p_3)^2 = (s_{13}^+ - s_{13}^-) \lambda_5 + s_{13}^-, \end{aligned} \quad (3.18)$$

$$\begin{aligned} s_{134}^\pm &= \frac{1}{2} \left[1 + \lambda_2 - \lambda_1(1 - \lambda_2)(1 - \sqrt{x_1})^2 + x_1(1 - \lambda_2) \pm (1 - \sqrt{x_1})(1 - \lambda_2) \sqrt{(1 - \lambda_1)((1 + \sqrt{x_1})^2 - \lambda_1(1 - \sqrt{x_1})^2)} \right], \\ s_{13}^\pm &= x_1 + \frac{1}{2} (1 - \sqrt{x_1}) [(\lambda_2(1 - \lambda_4) + \lambda_4)(1 + \sqrt{x_1} - \lambda_1(1 - \sqrt{x_1})) \\ &\quad + (\lambda_2(1 - \lambda_4) - \lambda_4)(1 - 2\lambda_3) \sqrt{(1 - \lambda_1)((1 + \sqrt{x_1})^2 - \lambda_1(1 - \sqrt{x_1})^2)}] \\ &\quad \pm 2(1 - \sqrt{x_1}) \sqrt{\lambda_2(1 - \lambda_3)\lambda_3(1 - \lambda_4)\lambda_4(1 - \lambda_1)((1 + \sqrt{x_1})^2 - \lambda_1(1 - \sqrt{x_1})^2)}. \end{aligned} \quad (3.19)$$

As it is expected if we put $x_1 = 0$ in (3.17) we get the formula (21) of [6] derived for massless particles in the final state.

In the Appendix B we consider the final state mass effects in some interesting phenomenological cases, providing fully integrated analytic formulae for the $1 \rightarrow 4$ processes.

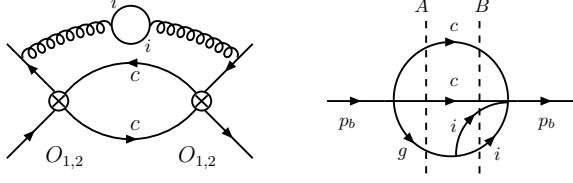


FIG. 1: Left: an example of an infrared divergent diagram contributing to the $B - \bar{B}$ process at NNLO. The virtual gluons are represented by curly lines. Right: one of the master integrals appearing after reduction of the left diagram with two cuts A and B (c - is a charm, g - is a gluon and i - is either charm or beauty or a light quark).

IV. CONTRIBUTION TO $B - \bar{B}$ MIXING AT $\mathcal{O}(\alpha_s^2)$

In this Section we illustrate how one can use formulae derived in previous sections for calculation of the absorptive part of a $B - \bar{B}$ mixing at $\mathcal{O}(\alpha_s^2)$, which contributes to the width differences $\Delta\Gamma$ and the CP-asymmetry.

An example of a diagram that contributes to the $B - \bar{B}$ mixing at $\mathcal{O}(\alpha_s^2)$ is shown in Fig. 1. This diagram contains both 3-particle and 4-particle cuts. The light quark masses we put equal to zero.

We introduce a small mass m_g for a gluon as an infrared regulator. In this way, in the matching one doesn't

need ϵ and ϵ^2 parts of NLO and LO Wilson coefficients correspondingly. Moreover, in the Effective Theory side the renormalization is much easier, because the $(1/\epsilon^n)$ terms appear only as UV singularities and so their cancellation can be tracked easier. Further in order to eliminate IR finite terms coming from the interference of IR $(1/\epsilon)$ with UV structures of $\mathcal{O}(\epsilon)$ a special class of operators ($\propto \epsilon$) has to be introduced in case of dimensional regularization. Whereas with introduction of a gluon mass those terms vanish automatically. This is the major difference in using dimensional regularization or gluon mass for IR singularities.

The reduction to the master integrals is done by means of a program FIRE [8].

Here we consider one of the most complicated master integrals (MI), shown on the right side of the Fig. 1. This MI has one three-particle cut and one four-particle cut. While in the case of 3-particle cut i on the right side of the Fig. 1 denotes either heavy m_c, m_b or light quarks, in the case of a 4-particle cut i stands for only light quarks. The corresponding cuts can be calculated analytically as an expansion over a gluon mass. The coefficient of a corresponding MI contains inverse masses of m_g up to the forth power.

The 3-particle cut (cut A) can be factorized into a product of a $q\bar{q}$ fermion loop and a 3-particle phase space, which is presented in the Appendix, formula (A.1). In the case of 4-particle cut we get:

$$\begin{aligned}
 r_B = & n_l \frac{m_b^2}{98304\pi^5} \left[16\pi^2(2 - 3x_c)x_c + 192x_c^2\text{Li}_2(\sigma) - 96(2 - x_c)x_c\text{Li}_2(\sigma^2) \right. \\
 & + 12\sqrt{1 - 4x_c}(2x_c + 1)(8\log(1 - \sigma) - 4\log(\sigma) - 5) \\
 & + 48\log(\sigma)(x_c^2\log(\sigma) - 4(2 - x_c)x_c\log(1 - \sigma) + (1 - x_c)^2 + 2(2 - x_c)x_c\log(1 - 4x_c)) \\
 & + x_g \cdot (-8\pi^2(5 - 6x_c) + 96(1 - 2x_c)\text{Li}_2(\sigma) + 48(3 - 2x_c)\text{Li}_2(\sigma^2) \\
 & + 24\log(\sigma)(4(3 - 2x_c)\log(1 - \sigma) + (1 - 2x_c)\log(\sigma) - 2(3 - 2x_c)\log(1 - 4x_c)) \\
 & + 8\sqrt{1 - 4x_c}(7\pi^2 - 6 - 48\log^2(1 - \sigma) - 3\log^2(\sigma) + 48\log(1 - \sigma)\log(\sigma)) \\
 & + x_g^2 \cdot \frac{4}{(1 - 4x_c)^{3/2}} (-28\pi^2x_c^2 - 3(6x_c^2 - 10x_c + 5) \\
 & + 12\log(\sigma)(1 - 2x_c - 2x_c^2 - \sqrt{1 - 4x_c}(1 - x_c) + x_c^2\log(\sigma)) \\
 & - 24\log(1 - \sigma)(1 - 2x_c - 2x_c^2 - 8x_c^2\log(1 - \sigma) + 8x_c^2\log(\sigma)) \\
 & + \log(x_g) \cdot (-24\sqrt{1 - 4x_c}(2x_c + 1) - 96x_c(1 - x_c)\log(\sigma) \\
 & + x_g(96(1 - x_c)\log(\sigma) + 96\sqrt{1 - 4x_c}(2\log(1 - \sigma) - \log(\sigma))) \\
 & + x_g^2 \frac{24}{(1 - 4x_c)^{3/2}} (1 - 2x_c - 2x_c^2 - 16x_c^2\log(1 - \sigma) + 8x_c^2\log(\sigma)) \\
 & \left. + \log^2(x_g) \cdot \left(-x_g 24\sqrt{1 - 4x_c} + x_g^2 \frac{48x_c^2}{(1 - 4x_c)^{3/2}} \right) \right] + \mathcal{O}(x_g^3),
 \end{aligned} \tag{4.1}$$

where n_l is the number of light quarks ($n_l = 3$), $x_c = (m_c/m_b)^2$, $x_g = (m_g/m_b)^2$ and $\sigma = \frac{1 - \sqrt{1 - 4x_c}}{1 + \sqrt{1 - 4x_c}}$.

In the expression (4.2) the infrared singularities appear as $\log^n(x_g)$, $n=1,2$. The infrared singularities of the diagram shown on the left side of Fig. 1, together with the $\log^n(x_g)$ -s of the other diagrams get canceled in the matching with the corresponding Effective Theory diagrams with Q , \tilde{Q}_S insertions [9, 10]. It must be mentioned also, that the diagrams both in the full and Effective Theory have to be renormalized. In addition, in the Effective Theory corresponding Evanescent operators have to be taken into account as well [11, 12].

V. CONCLUSIONS

To conclude, we have presented new formulae for computing $1 \rightarrow 3$ and $1 \rightarrow 4$ processes, with massive particles in the final states. We have also obtained a factorized formula for the 4 particle phase space with 1 massive and 3 massless particles in the final state. On the example of $B - \bar{B}$ mixing we demonstrate the capability of our technique for NNLO calculations, considering one of the most complicated master integrals. Further we study the final state mass effects in some interesting phenomeno-

logical cases providing fully integrated analytic formulae for 3 and 4 particle phase spaces.

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APPENDIX A: FULLY INTEGRATED 3-PARTICLE PHASE-SPACE

Here, as an application of the formula (2.4) we present fully integrated 3-particle phase space. For the case of $m_1 = m_2$, $m_3 = m_{light}$ ($m_{light} \ll m_{1,2}$), expanding (2.4) over m_{light}/m and ϵ and performing integration, we get in the \overline{MS} scheme

$$\begin{aligned}
 PS_3(m_1, m_1, m_{light}) = & \left(\frac{m^2}{\mu^2} \right)^{-2\epsilon} \frac{m^2}{128\pi^3} \left[\frac{1}{2} \sqrt{1-4x_1} (1+2x_1) + 2x_1 (1-x_1) \log(\sigma) \right. \\
 & + x_\ell (-2(1-x_1) \log(\sigma) + \sqrt{1-4x_1} (\log(x_\ell) - 4 \log(1-\sigma) + 2 \log(\sigma))) \\
 & + x_\ell^2 \frac{1}{2(1-4x_1)^{3/2}} (-1+2x_1+2x_1^2-4x_1^2 (\log(x_\ell) - 4 \log(1-\sigma) + 2 \log(\sigma))) \\
 & + \epsilon \cdot \left[-\frac{4\pi^2}{3} (1-x_1)x_1 + 4(1-x_1)x_1 (\text{Li}_2(\sigma) + \text{Li}_2(\sigma^2)) \right. \\
 & + (1+2x_1)\sqrt{1-4x_1} \left(\frac{13}{4} - \frac{1}{2} \log(1-4x_1) - 2 \log(1-\sigma) + \log(\sigma) \right) \\
 & - (1-x_1) \log(\sigma) (1-5x_1+6x_1 \log(1-4x_1) - 12x_1 \log(1-\sigma) + x_1 \log(\sigma)) \\
 & + x_\ell \left(\frac{4\pi^2}{3} (1-x_1) - 4(1-x_1) (\text{Li}_2(\sigma) + \text{Li}_2(\sigma^2)) \right. \\
 & + (1-x_1) \log(\sigma) (-6+6 \log(1-4x_1) - 12 \log(1-\sigma) + \log(\sigma)) \\
 & + \sqrt{1-4x_1} (4 \log(1-\sigma) (2 \log(1-\sigma) - 2 \log(\sigma) - 3) + (6 - \log(\sigma)) \log(\sigma) \\
 & + \log(1-4x_1) (4 \log(1-\sigma) - 2 \log(\sigma) - \log(x_\ell)) - \frac{1}{2} \log^2(x_\ell) + 3 \log(x_\ell) - 2\pi^2 + 2) \left. \right) \\
 & + x_\ell^2 \frac{1}{(1-4x_1)^{3/2}} \left(4\pi^2 x_1^2 + \frac{1}{4} (22x_1^2 - 14x_1 + 7) \right. \\
 & + \log(x_\ell) \left(x_1^2 \log(x_\ell) + \frac{1}{2} (-6x_1^2 + 2x_1 - 1) + 2x_1^2 \log(1-4x_1) \right) \\
 & - 2 \log(\sigma) (-x_1^2 \log(\sigma) + 2x_1^2 - 2x_1 + 1 - \sqrt{1-4x_1} (1-x_1)) \\
 & + \frac{1}{2} \log(1-4x_1) (-16x_1^2 \log(1-\sigma) + 8x_1^2 \log(\sigma) - 2x_1^2 - 2x_1 + 1) \\
 & \left. \left. + \log(1-\sigma) (4(2x_1^2 - 2x_1 + 1) - 16x_1^2 \log(1-\sigma) + 16x_1^2 \log(\sigma)) \right) \right] + \mathcal{O}(\epsilon^2, x_\ell^3) .
 \end{aligned} \tag{A.1}$$

For the case $m_1 > m_3$ and $m_2 = 0$ we have

$$\begin{aligned}
 PS_3(m_1, 0, m_3) = & \frac{m^2}{128\pi^3} \left[\frac{1}{2} (1 + x_1 + x_3) \sqrt{1 - 2(x_1 + x_3) + (x_1 - x_3)^2} \right. \\
 & - 2(x_1 + x_3 - 2x_1x_3) \coth^{-1} \left(\sqrt{\frac{1 - (\sqrt{x_1} - \sqrt{x_3})^2}{1 - (\sqrt{x_1} + \sqrt{x_3})^2}} \right) \\
 & + (x_1 - x_3) \left(2 \log \left(\sqrt{1 - (\sqrt{x_1} + \sqrt{x_3})^2} (\sqrt{x_1} - \sqrt{x_3}) + \sqrt{1 - (\sqrt{x_1} - \sqrt{x_3})^2} (\sqrt{x_1} + \sqrt{x_3}) \right) \right. \\
 & \left. \left. - \frac{1}{2} \log(16x_1x_3) \right) \right] + \mathcal{O}(\epsilon) .
 \end{aligned} \tag{A.2}$$

For $m_1 \gg m_3 = m_{light}$, expanding over m_{light}/m , we get also the terms proportional to ϵ^1

$$\begin{aligned}
 PS_3(m_1, 0, m_{light}) = & \left(\frac{m^2}{\mu^2} \right)^{-2\epsilon} \frac{m^2}{128\pi^3} \left[\frac{1}{2} (1 - x_1^2) + x_1 \log(x_1) \right. \\
 & + x_\ell ((1 - x_1) \log(x_\ell) - 2(1 - x_1) \log(1 - x_1) - x_1 \log(x_1)) + x_\ell^2 \left(\frac{1}{2} - \frac{1}{1 - x_1} \right) \\
 & + \epsilon \cdot \left[\frac{13}{4} (1 - x_1^2) - \frac{\pi^2 x_1}{3} + \frac{1}{2} x_1 (6 - x_1 - \log(x_1)) \log(x_1) \right. \\
 & \quad - 2 \log(1 - x_1) (1 - x_1^2 + x_1 \log(x_1)) - 2x_1 (\text{Li}_2(1 - x_1) - \text{Li}_2(x_1)) \\
 & + x_\ell ((1 - x_1) (6 \log^2(1 - x_1) - 6 \log(1 - x_1) + 3 \log(x_\ell) + 2) \\
 & \quad - \frac{1}{2} (1 - x_1) (4 \log(1 - x_1) \log(x_\ell) + \log^2(x_\ell)) - \pi^2 \left(1 - \frac{4x_1}{3} \right) \\
 & \quad - x_1 \left(-2 \log(1 - x_1) - \frac{\log(x_1)}{2} + 3 \right) \log(x_1) + 2 (\text{Li}_2(1 - x_1) - x_1 \text{Li}_2(x_1)) \Big) \\
 & \left. + x_\ell^2 \left(\frac{1 + x_1}{1 - x_1} \left(3 \log(1 - x_1) - \frac{\log(x_\ell)}{2} + \frac{7}{4} \right) - \frac{x_1 \log(x_1)}{1 - x_1} \right) \right] + \mathcal{O}(\epsilon^2, x_\ell^3) ,
 \end{aligned} \tag{A.3}$$

where the following notations were introduced

$$x_\ell = (m_{light}/m)^2, \quad x_{1,3} = (m_{1,3}/m)^2, \quad \sigma = \frac{1 - \sqrt{1 - 4x_1}}{1 + \sqrt{1 - 4x_1}} . \tag{A.4}$$

APPENDIX B: FULLY INTEGRATED 4-PARTICLE PHASE-SPACE

As an application of the formula (3.10) we give fully integrated 4-particle phase space. For the case when

$m_1 = m_2$, $m_3 = m_{light}$, $m_4 = 0$ ($m_{light} \ll m_{1,2}$), expanding (3.10) over m_{light}/m and ϵ and performing integration we get

$$\begin{aligned}
 PS_4(m_1, m_1, m_{light}, 0) = & \frac{m^4}{8192\pi^5} \left[-\frac{4\pi^2 x_1^2}{3} + \frac{1}{3} \sqrt{1 - 4x_1} (1 + 20x_1 + 12x_1^2) + 4x_1 (1 + x_1 - 2x_1^2) \log(\sigma) \right. \\
 & + 4x_1^2 \log(\sigma) (2 \log(1 - 4x_1) - 4 \log(1 - \sigma) + \log(\sigma)) - 16x_1^2 \text{Li}_2(-\sigma) \\
 & + x_\ell \cdot (-16x_1^2 \text{Li}_2(\sigma) + 8x_1 (2 - x_1) \text{Li}_2(\sigma^2) + 4 (x_1^2 - 1 + \sqrt{1 - 4x_1} (2x_1 + 1)) \log(\sigma) \\
 & - 4x_1^2 \log^2(\sigma) - \log(1 - \sigma) (8\sqrt{1 - 4x_1} (2x_1 + 1) - 16(2 - x_1)x_1 \log(\sigma)) \\
 & - 8(2 - x_1)x_1 \log(\sigma) \log(1 - 4x_1) + \log(x_\ell) (8(1 - x_1)x_1 \log(\sigma) + 2\sqrt{1 - 4x_1} (2x_1 + 1)) \\
 & \left. - \frac{4}{3} \pi^2 (2 - 3x_1)x_1 + 3\sqrt{1 - 4x_1} (2x_1 + 1) \right) \\
 & + x_\ell^2 \cdot (-4(1 - x_1) \log(\sigma) - \sqrt{1 - 4x_1} (3 + 8 \log(1 - \sigma) - 4 \log(\sigma) - 2 \log(x_\ell))) \Big] + \mathcal{O}(\epsilon, x_\ell^3) .
 \end{aligned} \tag{B.1}$$

For $m_1 > m_3$, $m_2 = m_4 = 0$ we have

$$\begin{aligned}
 PS_4(m_1, 0, m_3, 0) = & \frac{m^4}{8192\pi^5} \left[4(x_1^2 - x_3^2) \tanh^{-1} \left(\sqrt{\frac{1 - (\sqrt{x_1} + \sqrt{x_3})^2}{1 - (\sqrt{x_1} - \sqrt{x_3})^2}} \frac{\sqrt{x_1} - \sqrt{x_3}}{\sqrt{x_1} + \sqrt{x_3}} \right) \right. \\
 & + \frac{1}{3} \sqrt{1 - 2(x_1 + x_3) + (x_1 - x_3)^2} (1 + x_1^2 + x_3^2 + 10(x_1 + x_3 + x_1 x_3)) \\
 & + 2(x_1 - x_3) \log(4x_1) + (x_1^2(1 - 2x_3) + 2x_3(1 - x_1 x_3) + x_3^2 + 4x_1 x_3 \log(x_3)) \\
 & \times \left(\log(4x_1 x_3) - 2 \log \left(1 - x_1 - x_3 + \sqrt{1 - 2(x_1 + x_3) + (x_1 - x_3)^2} \right) \right) \\
 & - 4(x_1 - x_3) \log \left(1 + x_1 - x_3 + \sqrt{1 - 2(x_1 + x_3) + (x_1 - x_3)^2} \right) \\
 & - 8x_1 x_3 \text{Li}_2 \left(\frac{2x_1}{x_1 + x_3 - 1 - \sqrt{1 - 2(x_1 + x_3) + (x_1 - x_3)^2}} \right) \\
 & \left. + 8x_1 x_3 \text{Li}_2 \left(\frac{x_1 + x_3 - 1 - \sqrt{1 - 2(x_1 + x_3) + (x_1 - x_3)^2}}{2x_3} \right) \right] + \mathcal{O}(\epsilon) .
 \end{aligned} \tag{B.2}$$

For $m_1 \gg m_3 = m_{\text{light}}$, expanding the expression (B.2) over m_3 , we get

$$\begin{aligned}
 PS_4(m_1, 0, m_{\text{light}}, 0) = & \frac{m^4}{8192\pi^5} \left[\frac{1}{3} (1 - x_1) (x_1^2 + 10x_1 + 1) + 2x_1 (x_1 + 1) \log(x_1) \right. \\
 & + x_\ell \left((1 - x_1^2) (3 - 4 \log(1 - x_1) + 2 \log(x_\ell)) + 2x_1 \log(x_1) (-x_1 - 4 \log(1 - x_1) + 2 \log(x_\ell)) \right. \\
 & \left. \left. - 8x_1 \text{Li}_2(1 - x_1) \right) \right. \\
 & \left. + x_\ell^2 (- (1 - x_1) (3 + 4 \log(1 - x_1) - 2 \log(x_\ell)) - 2x_1 \log(x_1)) \right] + \mathcal{O}(\epsilon, x_\ell^3) .
 \end{aligned} \tag{B.3}$$

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